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SUDDEN APPEARANCE OF A CRACK IN A BENT PLATE†

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Abstract—This paper is concerned with the determination of the transient stress state at the site in a bent plate where a through crack of finite length has appeared suddenly. Such a phenomenon can be easily visualized as the type of damage caused by projectile penetration. Mindlin's equations of flexural motion are used so that the three natural boundary conditions can be satisfied on the crack surfaces whereas the approximate Kirchhoff condition in the classical theory leads to unrealistic results near the crack. A set of dual integral equations is derived for this problem and solved numerically. The moment intensity factor is found to increase monotonically with time and is always lower than the static limit. Its amplitude is a function of the plate thickness to crack length ratio. This is different from the in-plane stretching case where the dynamic stress-intensity factor rises above the static limit very quickly before it decays.

INTRODUCTION

A POSSIBLE failure mode of plate-like structures when subjected to ballistic impact is penetration of the projectile accompanied by rapid crack propagation. Depending on the speed of the projectile, the thickness of the target material, and other factors, the crack may reach a subcritical length and arrest itself. In such a case, the problem may be modeled by the sudden appearance of a through crack.

Transient stress analysis dealing with the state of affairs near the crack tip region has been carried out by a number of past investigators. Their contributions are reviewed and mentioned in [1, 2]. Among the problems solved are shear and dilatational waves passing by a plane [3–5] or penny-shaped [6, 7] crack. The general conclusion is that the spatial distribution of the dynamic stresses around the crack edge remained the same as the static case while the amplitude of the local stress field varied with time reaching a peak approximately 20–30 per cent greater than the corresponding static value and subsequently oscillating about that value. The problem of flexural waves scattering at a crack was solved by Sih and Loeber [8] who considered only the steady state situation.

The present analysis is concerned with the effect of bending on the transient stresses near the tip of the crack. The problem will be examined through the theory of flexural motion of plates developed by Mindlin [9] in which the three physical boundary conditions of applied bending moments and zero twisting moment and transverse shear stress can be satisfied individually at the crack edge. The procedure will be to apply transform

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techniques to Mindlin's equations of motion in order to obtain a set of dual integral equations from which a numerical solution can be obtained.

EQUATIONS OF FLEXURAL MOTION OF PLATES

The assumptions on the displacement vector are that the in-plane components, u and v, are linearly dependent on the out-of-plane z coordinate, and that the out-of-plane component, w, is independent of z. That is, the rectangular components of displacement assume the form

$$
\begin{aligned}
u(x, y, z, t) &= z\psi_1(x, y, t) \\
v(x, y, z, t) &= z\psi_2(x, y, t) \\
w(x, y, z, t) &= \psi_3(x, y, t).
\end{aligned} \tag{1}
$$

The functions ψ_1, ψ_2 and ψ_3 are called plate-displacement components and are illustrated together with the general plate geometry in Fig. 1. The bending and twisting moments and transverse shearing forces (per unit length) are defined, in terms of the stress components, as

$$(M_x, M_y, H_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z \, \mathrm{d}z \tag{2}$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) \, \mathrm{d}z \tag{3}$$

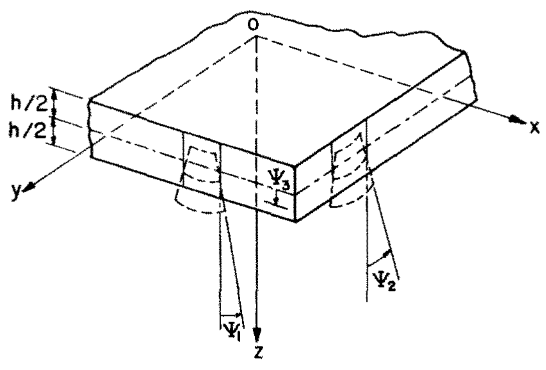

FIG. 1. General plate geometry and displacements.

where h is the thickness of the plate. The plate-stress components just defined are shown acting on the element of the plate in Fig. 2.

Through use of the stress–strain and constitutive equations of classical three-dimensional elasticity, the following relations are obtained between the plate-stress and plate-

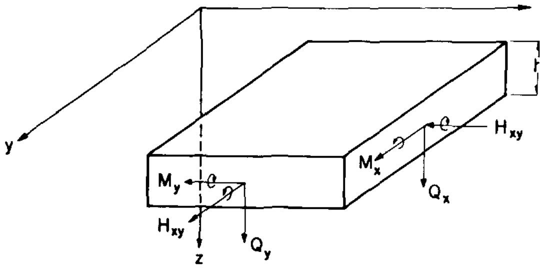

FIG. 2. Moments and shearing forces on plate element.

displacement components,

$$
\begin{aligned}
M_x &= D\left(\frac{\partial\psi_1}{\partial x}+\nu\frac{\partial\psi_2}{\partial y}\right)\\
M_y &= D\left(\frac{\partial\psi_2}{\partial y}+\nu\frac{\partial\psi_1}{\partial x}\right)\\
H_{xy} &= \frac{1-\nu}{2}D\left(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x}\right)\\
Q_x &= \kappa^2\mu h\left(\frac{\partial\psi_3}{\partial x}+\psi_1\right)\\
Q_y &= \kappa^2\mu h\left(\frac{\partial\psi_3}{\partial y}+\psi_2\right)
\end{aligned}
\tag{4}
$$

with ν and μ being Poisson's ratio and the shear modulus respectively and where the shear coefficient κ^2 assumes the value $\pi^2/12$. The flexural rigidity of the plate, D, is given by

$$D = \frac{\mu h^3}{6(1-\nu)}. \tag{5}$$

Making use of equations (1–4), the equations of motion of three-dimensional elasticity theory may be converted to the following equations for plate-displacements.

$$
\begin{aligned}
\frac{D}{2}\left[(1-\nu)\nabla^2\psi_1+(1+\nu)\frac{\partial\Phi}{\partial x}\right]-\kappa^2\mu h\left(\psi_1+\frac{\partial\psi_3}{\partial x}\right) &= \frac{\rho h^3}{12}\frac{\partial^2\psi_1}{\partial t^2}\\
\frac{D}{2}\left[(1-\nu)\nabla^2\psi_2+(1-\nu)\frac{\partial\Phi}{\partial y}\right]-\kappa^2\mu h\left(\psi_2+\frac{\partial\psi_3}{\partial y}\right) &= \frac{\rho h^3}{12}\frac{\partial^2\psi_2}{\partial t^2}\\
\kappa^2\mu h(\nabla^2\psi_3+\Phi)+q &= \rho h\frac{\partial^3\psi_3}{\partial t^2}
\end{aligned}
\tag{6}
$$

where

$$\Phi = \frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}. \tag{7}$$

The effect of the normal pressure q, will not be considered here and therefore q will be set equal to zero. The parameter ρ is the mass density and ∇^2 is the Laplacian operator.

FORMULATION OF PROBLEM

The specific problem to be considered here is that of sudden appearance of uniform bending moment on the surface of a crack of length $2a$ lying on the x-axis of an infinite plate (see Fig. 3). Since this problem is symmetric with respect to the x-axis, it will be sufficient to consider only the region of the plate where $y > 0$. As is usual in problems of this type where geometric singularities are involved, mixed boundary conditions are imposed on the x-axis. These conditions are

$$\begin{aligned} &Q_y(x, o, t) = 0 && \text{for} \quad 0 \leq |x| < \infty \\ &H_{xy}(x, o, t) = 0 && \text{for} \quad 0 \leq |x| < \infty \\ &M_y(x, o, t) = -M_0 H(t) && \text{for} \quad |x| < a \\ &\psi_2(x, o, t) = 0 && \text{for} \quad |x| > a. \end{aligned} \tag{8}$$

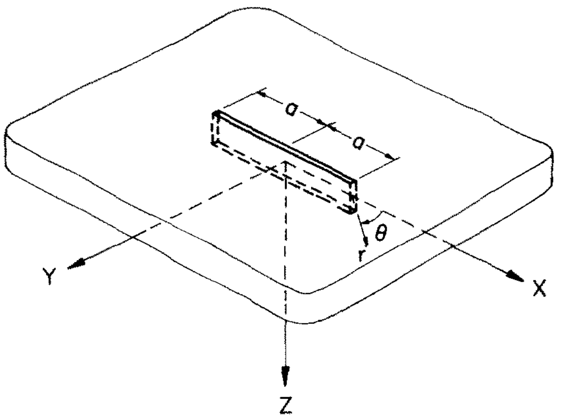

FIG. 3. Geometry of through crack in plate.

In addition, the condition on displacement at infinity is

$$\lim_{x^2+y^2 \to \infty} [\psi_1(x, y, t), \psi_2(x, y, t), \psi_3(x, y, t] = 0$$

and the initial conditions are taken to be zero. The governing field equations for the plate may now be solved subject to the preceding conditions.

Recall that the Laplace transform pair is

$$f^*(p) = \int_0^\infty f(t) \exp(-pt) \, dt \tag{9}$$

$$f(t) = \frac{1}{2\pi i} \int_{B_r} f^*(p) \exp(pt) \, dp \tag{10}$$

where the second integral is over the Bromwich [10] path. The result of applying equation (9) to equation (6) is

$$\frac{D}{2}\left[(1-\nu)\nabla^2\psi_1^* + (1+\nu)\frac{\partial\Phi^*}{\partial x}\right] - \kappa^2\mu h\left(\psi_1^* + \frac{\partial\psi_3^*}{\partial x}\right) = \frac{\rho h^3}{12}p^2\psi_1^* \tag{11}$$

$$\frac{D}{2}\left[(1-\nu)\omega^2\psi_2^* + (1-\nu)\frac{\partial\Phi^*}{\partial y}\right] - \kappa^2\mu h\left(\psi_2^* + \frac{\partial\psi_3^*}{\partial y}\right) = \frac{\rho h^3}{12}p^2\psi_2^* \tag{12}$$

$$\kappa^2\mu h(\nabla^2\psi_3^* + \Phi^*) = \rho h p^2\psi_3^*. \tag{13}$$

Note that the reduction of the preceding equations to a set of homogeneous ordinary differential equations necessitates the introduction of the Fourier cosine and sine transforms. The Fourier cosine transform pair is

$$\bar{f}(\alpha) = \int_0^\infty f(x)\cos(\alpha x)\,\mathrm{d}x, \qquad f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}(\alpha)\cot(\alpha x)\,\mathrm{d}\alpha$$

and the Fourier sine transform pair is

$$\hat{\mathbf{f}}(\alpha) = \int_0^\infty f(x)\sin(\alpha x)\,\mathrm{d}x, f(x) = \frac{2}{\pi}\int_0^\infty \hat{\mathbf{f}}(\alpha)\sin(\alpha x)\,\mathrm{d}\alpha$$

By noting the symmetry properties of the boundary conditions and by consideration of equations (5), it is easily seen that ψ_2 and ψ_3 are even in x and ψ_1 is odd in x. That is,

$$\psi_1(x, y, t) = -\psi_1(-x, y, t), \qquad \psi_2(x, y, t) = \psi_2(-x, y, t)$$
$$\psi_3(x, y, t) = \psi_3(-x, y, t).$$

Thus, the Fourier sine transform is applied to equation (11), and the Fourier cosine transform is applied to equations (12) and (13) with the result

$$\frac{D}{2}(1-\nu)\frac{\partial^2\hat{\Psi}_1^*}{\partial y^2} - \left(\frac{\rho h^3}{12}p^2 + D\alpha^2 + \kappa^2\mu h\right)\hat{\Psi}_1^* - \frac{D}{2}(1+\nu)\alpha\frac{\partial\bar{\Psi}_2^*}{\partial y} + \kappa^2\mu h\alpha\bar{\Psi}_3^* = 0$$

$$\frac{D}{2}(1+\nu)\alpha\frac{\partial\hat{\Psi}_1^*}{\partial y} + D\frac{\partial^2\bar{\Psi}_2^*}{\partial y^2} - \left(\frac{\rho h^3 p^2}{12} + \frac{D}{2}(1-\nu)\alpha^2 + \kappa^2\mu h\right)\bar{\Psi}_2^* - \kappa^2\mu h\frac{\partial\bar{\Psi}_3^*}{\partial y} = 0 \tag{14}$$

$$\kappa^2\mu h\alpha\hat{\Psi}_1^* + \kappa^2\mu h\frac{\partial\bar{\Psi}_2^*}{\partial y} + \kappa^2\mu h\frac{\partial^2\bar{\Psi}_3^*}{\partial y^2} - (\alpha^2\kappa^2\mu h + \rho h p^2)\bar{\Psi}_3^* = 0.$$

The preceding set of coupled ordinary differential equations may be solved in the usual manner with the following result

$$\begin{aligned}
\hat{\Psi}_1^* &= -a_1^*(\alpha, p)(\sigma_1 - 1)\alpha\exp(-\gamma_1 y) - a_2^*(\alpha, p)(\sigma_2 - 1)\alpha\exp(-\gamma_2 y) \\
&\quad - a_3^*(\alpha, p)\gamma_3\exp(-\gamma_3 y) \\
\bar{\Psi}_2^* &= -a_1^*(\alpha, p)(\sigma_1 - 1)\gamma_1\exp(-\gamma_1 y) - a_2^*(\alpha, p)(\sigma_2 - 1)\gamma_2\exp(-\gamma_2 y) \\
&\quad - \alpha a_3^*(\alpha, p)\exp(-\gamma_3 y) \\
\bar{\Psi}_3^* &= a_1^*(\alpha, p)\exp(-\gamma_1 y) + a_2^*(\alpha, p)\exp(-\gamma_2 y)
\end{aligned} \tag{15}$$

where

$$\begin{aligned} \gamma_1 &= \sqrt{[\alpha^2+(\beta_1^2/a^2)]} \\ \gamma_2 &= \sqrt{[\alpha^2+(\beta_2^2/a^2)]} \\ \gamma_3 &= \sqrt{[\alpha^2+(\beta_3^2/a^2)]} \end{aligned} \tag{16}$$

$$(\sigma_1, \sigma_2) = \frac{2}{1-\nu}(\beta_2^2, \beta_1^2)\beta_3^{-2} \tag{17}$$

and

$$\begin{aligned} 2(\beta_1^2, \beta_2^2) &= \frac{p^2}{\omega_0^2(h/a)^2}\left\{\frac{1}{S}+12\pm\left[\left(\frac{1}{S}-12\right)^2-\frac{48}{S}\left(\frac{\omega_0}{p}\right)^2\right]^{\frac{1}{2}}\right. \\ \beta_3^2 &= \frac{\pi^2}{(h/a)^2}\left[\left(\frac{p}{\omega_0}\right)^2+1\right]. \end{aligned} \tag{18}$$

The quantity S stands for

$$S = \frac{2}{\pi^2(1-\nu)}$$

and ω_0 is the cut-off frequency

$$\omega_0 = \frac{\pi}{h}c_2$$

with $c_2 = (\mu/\rho)^{\frac{1}{2}}$ being the shear wave velocity in an infinite solid.

The expressions for the plate-displacement components simplify considerably through introduction of the boundary conditions on Q_y and H_{xy} in equation (8). That is, the equations,

$$\begin{aligned} Q_y^* &= \kappa^2\mu h\left(\frac{\partial\psi_3^*}{\partial y}+\psi_2^*\right) = 0 \qquad \text{for} \quad y = 0, \\ H_{xy}^* &= \frac{1-\nu}{2}D\left(\frac{\partial\psi_1^*}{\partial y}+\frac{\partial\psi_2^*}{\partial x}\right) = 0 \quad \text{for} \quad y = 0, \end{aligned} \tag{19}$$

allow the functions $a_i^*(\alpha, p)$ $(i = 1, 2, 3)$ to be written

$$\begin{aligned} a_1^*(\alpha, p) &= \frac{A^*(\alpha, p)}{(\beta_1^2-\beta_2^2)}\left[\frac{f_1(\alpha)}{(1-\nu)\alpha^2a^2+\beta_2^2}\right] \\ a_2^*(\alpha, p) &= -\frac{A^*(\alpha, p)}{(\beta_1^2-\beta_2^2)}\left[\frac{f_2(\alpha)}{(1-\nu)\alpha^2a^2+\beta_2^2}\right] \\ a_3^*(\alpha, p) &= \frac{\alpha a^2A^*(\alpha, p)}{(\beta_2^2-\beta_2^2)}(1-\nu)(\sigma_2-\sigma_1) \end{aligned} \tag{20}$$

where

$$f_1(\alpha) = (\alpha^2+\beta_1^2/a^2)^{-\frac{1}{2}}[(1-\nu)\alpha^2 a^2+\beta_1^2]^2$$
$$f_2(\alpha) = (\alpha^2+\beta_2^2/a^2)^{-\frac{1}{2}}[(1-\nu)\alpha^2 a^2+\beta_2^2]^2 \tag{21}$$

The solution of the problem now depends on the evaluation of $A^*(\alpha, p)$. This may be accomplished through application of the remaining two boundary conditions of equation (8). The result is the following set of dual integral equations:

$$\frac{2}{\pi}\int_0^\infty F(\alpha, p)A^*(\alpha, p)\cos(\alpha x)\,d\alpha = \frac{M_0}{pD}|x| < a$$
$$\int_0^\infty A^*(\alpha, p)\cos(\alpha x)\,d\alpha = 0|x| > a \tag{22}$$

where

$$F(\alpha, p) = \frac{1}{(\beta_1^2-\beta_2^2)}\left[(1-\sigma_1)\frac{f_1(\alpha)}{a^2}-(1-\sigma_2)\frac{f_2(\alpha)}{a^2}\right.$$
$$\left.-(1-\nu)^2(\sigma_2-\sigma_1)\alpha^2 a^2\sqrt{[\alpha^2+(\beta_3/a)^2]}\right] \tag{23}$$

The procedure by which the dual integral equations are reduced to the solution of a Fredholm integral equation is the same as that used for the in-plane case [2] and will not be repeated here. The result of this procedure is that $A^*(\alpha, p)$ may be written

$$A^*(s, p) = \frac{\pi a^2 M_0}{pD(1-\nu^2)}\int_0^1 \Phi^*(\xi, p)J_0(s\xi a)\sqrt{(\xi)}\,d\xi \tag{24}$$

where J_0 is the zero-order Bessel function of the first kind and $\Phi^*(\xi, p)$ is the solution to the following Fredholm integral equation:

$$\Phi^*(\xi, p)-\int_0^1 \Phi^*(\eta, p)K(\xi, \eta)\,d\eta = \sqrt{(\xi)},\ \xi < 1 \tag{25}$$

whose kernel being symmetric in ξ and η is

$$K(\xi, \eta) = \frac{2\sqrt{(\xi\eta)}}{(1-\nu^2)}\int_0^\infty sf(s, p)J_0(s\eta)J_0(s\xi)\,ds \tag{26}$$

with

$$f(s, p) = \tfrac{1}{2}(1-\nu^2)-\frac{aF(s/a, p)}{s}.$$

As noted previously [2] rapid convergence of the infinite integral is facilitated if a function $g(s, p)$ is defined to be

$$g(s, p) = f(s, p)+\frac{H}{s^2+E^2}$$

where

$$H = \frac{1}{8(h/a)^2 S}\{[1+2\nu+3\nu^2](p^2/\omega_0^2)+4\}$$

$$E^2 = \frac{\{(1-\nu)[5\nu^2+2\nu+1](p/\omega_0)^2[(p/\omega_0)^2-12S]+8[(p/\omega_0)^2+1]^2\}}{16(1-\nu)(h/a)^2S^2H}$$

Then, noting that

$$\int_0^\infty \frac{Hs}{s^2+E^2}J_0(s\eta)J_0(s\xi)\,\mathrm{d}s = HI_0(E\xi)K_0(E\eta), \qquad 0 < \xi \le \eta$$

the kernel becomes

$$K(\xi,\eta) = \frac{2\sqrt{(\xi\eta)}}{(1-\nu^2)}\left\{-HI_0(\xi E)K_0(\eta E)+\int_0^\infty sg(s,p)J_0(s\eta)J_0(s\xi)\,\mathrm{d}s\right\}, \qquad 0 < \xi \le \eta \quad (27)$$

TIME DEPENDENT MOMENT DISTRIBUTION

The solution of the Fredholm integral equation for Φ^* completely determines the Laplace transforms of the plate-stress components. Since the three components of moment are expected to possess a square root singularity near the crack tip these will be discussed in detail. Making use of equations (14) and (15) and the definition of the Fourier transform pairs, these may be written

$$\begin{aligned}
M_x^* &= \frac{2}{\pi}D\int_0^\infty [(\sigma_1-1)[\nu\gamma_1^2-\alpha^2]a_1^*(s,p)\exp(-\gamma_1 y)+(\sigma_2-1)[\nu\gamma_2^2-\alpha^2]a_2^*(s,p)\exp(-\gamma_2 y)\\
&\quad -(1-\nu)\gamma_3\alpha\exp(-\gamma_3 y)]\cos(\alpha x)\,\mathrm{d}\alpha\\
M_y^* &= \frac{2}{\pi}D\int_0^\infty [(\sigma_1-1)[\gamma_1^2-\nu\alpha^2]a_1^*(s,p)\exp(-\gamma_1 y)+(\sigma_2-1)[\gamma_2^2-\nu\alpha^2]a_2^*(s,p)\exp(-\gamma_2 y)\\
&\quad +(1-\nu)\gamma_3\alpha\exp(-\gamma_3 y)]\cos(\alpha x)\,\mathrm{d}\alpha\\
H_{xy}^* &= \frac{(1-\nu)}{\pi}D\int_0^\infty [2(\sigma_1-1)\alpha\gamma_1 a_1^*(s,p)\exp(-\gamma_1 y)+2(\sigma_2-1)\alpha\gamma_2 a_2^*(s,p)\exp(-\gamma_2 y)\\
&\quad +(\gamma_3^2+\alpha^2)a_3^*(s,p)\exp(-\gamma_3 y)]\sin(\alpha x)\,\mathrm{d}\alpha.
\end{aligned} \quad (28)$$

In Ref. [2], the inverse Laplace transforms of the stress expressions were obtained through application of the Cagniard–DeHoop inversion technique. However, as this is not possible for the present case, an alternative technique of asymptotic expansion of the stress-field near the crack tip before inversion will be used [7]. Observe that the infinite integrals in equations (28) are convergent everywhere except at the singular points, i.e., the crack tips. To obtain the solution near the crack tip, it is necessary to evaluate the unbounded portions of these integrals in the neighborhood of the singular points. Noting that the integrands are finite and continuous for any given value of s, the divergence of the integrals near the crack tip must be due to behavior as $s \to \infty$. Hence, the terms that give rise to unbounded plate-stresses correspond to those parts of the integrand that are dominant for large values of s, and they will be isolated in the work to follow.

If the expression for $A^*(\alpha, p)$ is integrated by parts so that

$$A^*(s,p) = \frac{\pi a M_0}{pD(1-\nu^2)s}\left[\Phi^*(1,p)J_1(sa) - \int_0^1 \frac{d}{d\xi}\left[\frac{\Phi^*(\xi,p)}{\sqrt{\xi}}\right] J_1(sa\xi)\sqrt{(\xi)}\, d\xi\right], \tag{29}$$

it can be seen that the singular part of the solution depends only on the first term on the right. Then making use of equations (20), the integrands of equations (28) may be expanded for large s, such that the result of retaining only the highest order term is

$$M_x^* = M_0 a\Phi^*(1,p)\int_0^\infty (sy-1)\exp(-sy)J_1(sa)\cos(sx)\, ds + \dots$$

$$M_y^* = -M_0 a\Phi^*(1,p)\int_0^\infty (sy+1)\exp(-sy)J_1(sa)\cos(sx)\, ds + \dots \tag{30}$$

$$H_{xy}^* = -M_0 a\frac{\Phi^*(1,p)}{p}\int_0^\infty sy\exp(-sy)J_1(sa)\sin(sx)\, ds + \dots$$

Making use of well-known Bessel integral identities and the Laplace inversion theorem, the following results are obtained for the plate stresses near the crack tip:

$$M_x = \frac{M_0\sqrt{a}}{\sqrt{(2r)}}N(t)\cos\left(\frac{\theta}{2}\right)\left\{1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right\} + \dots$$

$$M_y = \frac{M_0\sqrt{a}}{\sqrt{(2r)}}N(t)\cos\left(\frac{\theta}{2}\right)\left\{1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right\} + \dots \tag{31}$$

$$H_{xy} = \frac{M_0\sqrt{a}}{\sqrt{(2r)}}N(t)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) + \dots$$

where the polar coordinates r and θ are shown in Fig. 3 and the function $N(t)$ is the inverse Laplace transform of $[\Phi^*(1, p)/p]$. The moments near the crack tip possess the inverse square-root of r singularity. Then, following Sih and Loeber [8], a dynamic moment intensity factor may be defined here as

$$K_1(t) = M_0\sqrt{(a)}N(t)$$

and the plate-stresses near the crack tip may be written

$$M_x = \frac{K_1(t)}{\sqrt{(2r)}}\cos\left(\frac{\theta}{2}\right)\left\{1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right\} + \dots$$

$$M_y = \frac{K_1(t)}{\sqrt{(2r)}}\cos\left(\frac{\theta}{2}\right)\left\{1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right\} + \dots$$

$$H_{xy} = \frac{K_1(t)}{\sqrt{(2r)}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) + \dots$$

where the expressions for the shear resultants Q_x and Q_y are not presented as they are finite everywhere. The preceding expressions for the moments are identical to the expressions obtained for σ_x, σ_y and τ_{xy} in the in-plane extension problem with the only difference depending on the behavior of $K_1(t)$.

In order to evaluate $K_1(t)$ it is necessary first to numerically solve the Fredholm integral equation (25) for $\Phi^*(\xi, p)$. The results of this are presented in Fig. 4 where values of $\Phi^*(1, p)$ as a function of (c_2/pa) are plotted for two values of (h/a) and for Poisson's ratio equal to 0·3. Since only the numerical values of $[\Phi^*(1, p)/p]$ are obtained, the inverse Laplace transform of this function, $N(t)$, must be obtained numerically.

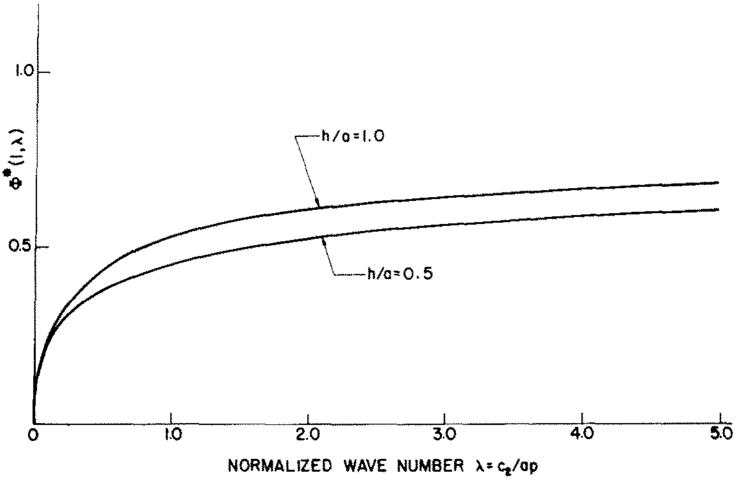

FIG. 4. Solutions to Fredholm integral equation.

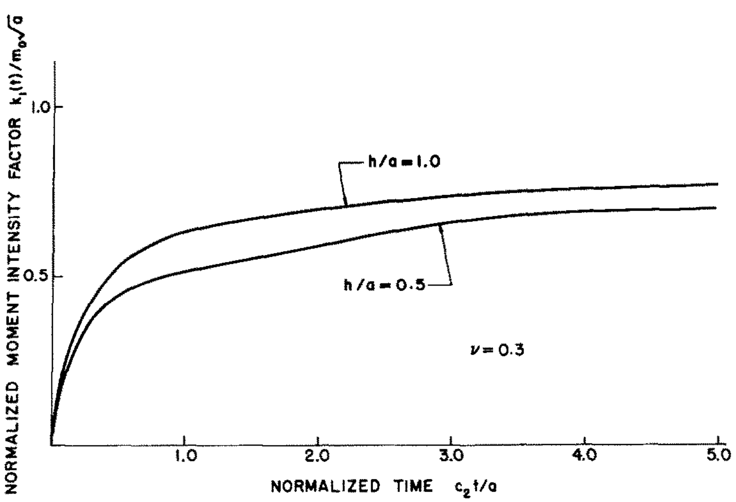

FIG. 5. Moment intensity factor as function of time.

The behavior of $K_1(t)$ that may thus be described is quite different from the case of uniform in-plane extension. Referring to Fig. 5, when the ratio of plate thickness to crack length, h/a, is equal to one, the dynamic moment–intensity factor asymptotically approaches the static value of $K_1(t) = 0{\cdot}75\, M_0\sqrt{a}$ with little or no oscillation about that value. It should be noted that this result is in agreement with Sih and Loeber [8] who, for the same geometric and material parameters, showed that the steady-state dynamic moment-intensity factor decreases from the static value as the frequency increases. Figure 5 also shows that $K_1(t)$ has the same behavior for a value of h/a equal to 0·5. The static limit in this case is $0{\cdot}70\, M_0\sqrt{a}$ which is in agreement with the result reported by Hartranft and Sih [11].

REFERENCES

[1] G. C. Sih, Some elastodynamic problems of cracks. *Int. J. Fracture Mech.* **4**, 51 (1968).
[2] G. C. Sih, G. T. Embley and R. S. Ravera, Impact response of a finite crack in plane extension. *Int. J. Solids Struct.* **8**, 977 (1972).
[3] J. D. Achenbach, Crack propagation generated by a horizontally polarized shear wave. *J. Mech. Phys. Solids* **17**, 177 (1969).
[4] R. J. Ravera and G. C. Sih, Transient analysis of stress waves around cracks under antiplane strain. *J. Acoust. Soc. Am.* **47**, 875 (1970).
[5] S. A. Thau and T. H. Lu, Transient stress-intensity factors for a finite crack in an elastic solid caused by a dilatational wave. *Int. J. Solids Struct.* **7**, 731 (1971).
[6] G. C. Sih and G. T. Embley, Sudden twisting of a penny-shaped crack. *J. appl. Mech.* **39**, 395 (1971).
[7] G. T. Embley and G. C. Sih, Response of a penny-shaped crack to impact waves. *Proc. 12th Midwestern Mech. Conf.* **6**, 473 (1971).
[8] G. C. Sih and J. F. Loeber, Flexural waves scattering at a through crack in an elastic plate. *J. Engng. Fracture Mech.* **1**, 369 (1968).
[9] R. D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic elastic plates. *J. appl. Mech.* **18**, 31 (1951).
[10] G. Doetsch, *Guide to the application of Laplace transformations.* Van Nostrand (1961).
[11] R. J. Hartranft and G. C. Sih, Effect of plate thickness on the bending stress distribution around through cracks. *J. Math. Phys.* **47**, 276 (1968).

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Абстракт—Работа занимается определением нестационарного напряженного состояния в этом месте изгибаемой пластинки, где появилась внезапно трещина насквозь, конечной длины. Такое явление можно легко мысленно представить себе в качестве повреждения, вызыванного проникновением снаряда. Используются уравнения Миндлина для изгиблого движения и, затем, удовлетворяются трме естественным граничным условиям на поверхностях трещины, тогда как приближенное условие Кирхгоффа приводит к нереалистическим результатам близи трещины. для этой задачи выводится система дуальных интегральных уравнений и решается численно. Укаеано, что фактор интенсивности момента увеличивается монотонно во времени и всегда является низшим по сравнению со статическим пределом. Его амплитуда оказывается функцией соотношения толщины пластинки к длине трещины. Это явление отличается от случая растяжения в плоскости, в котором фактор интенсивности динамических напряжений повышается очень быстро над статический предел, раньше разрушения.